# The rotationally symmetric flow of a viscous fluid in the presence of an infinite rotating disk

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The flow produced by an infinite rotating disk when the fluid at infinity is in a state of solid rotation is investigated numerically. When the fluid at infinity is rotating in the same sense as the disk, physically acceptable solutions exist in all cases. When the fluid at infinity is rotating in the opposite sense to that of the disk, the only physically acceptable solutions appear to be those in which there is a uniform suction present acting through the disk.

## 1. Introduction

The steady flow of an incompressible viscous liquid, due to an infinite rotating disk, was first discussed by von Kármán (1921). The liquid occupies the semi-infinite region on one side of the disk and the motion is rotationally symmetric. The effect of the disk is to throw the fluid near its surface radially outwards, and this in turn induces an axial inflow. The main interest in this problem is that, by virtue of assumptions about the velocity components, the Navier–Stokes equations reduce to a set of ordinary, non-linear differential equations in a single independent variable.

These equations are, in fact, the boundary-layer equations for the problem, since the terms which are ordinarily omitted in boundary-layer theory vanish identically. Numerical integration of this set of equations thus yields an exact solution of the Navier–Stokes equations. Von Kármán obtained an approximate solution to the problem using the integral method he invented, while Cochran (1934) corrected his solution and then calculated more accurate values by numerical integration of the equations. Bödewadt (1940) solved numerically the related problem of the flow produced over an infinite stationary plane in fluid which is rotating with uniform angular velocity at an infinite distance from the plane. Both these flows are particular cases of a general family of rotationally symmetric flows which were described qualitatively by Batchelor (1951). In these flows the fluid at infinity has an arbitrary uniform angular velocity about the axis of rotation of the disk, leading to a one-parameter family of solutions.

Stewartson (1953) has also considered this problem and has obtained approximate solutions in certain cases, while Squire (1953) linearized the equations for

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perturbations about a state of solid rotation. A complete discussion of these and other related papers is to be found in a review article by Moore (1956).

Recently Fettis (1955) has devised an iterative process to obtain approximate solutions for several cases in which the fluid at infinity is rotating in the same sense as the disk. However, when the fluid at infinity is rotating in the opposite sense to that of the disk, the process diverges. It was therefore decided to investigate numerically the general set of equations, and to find out for what cases physically acceptable solutions exist. All the numerical integrations have been performed on the Pegasus digital computer at the University of Southampton.

It should perhaps be emphasized that the present work, like the above-mentioned papers, refers to an infinite disk and is further based on the assumption of a similarity solution. Consequently, any edge effects for a disk of *finite* radius are automatically ignored, and in cases when the flow in the boundary layer is radially inwards the question of the range of validity of the similarity solution has still to be settled; a discussion of this point has been given by Stewartson (1957). When the flow in the boundary layer is radially outwards the similarity solution is simply the first approximation in an expansion of the dependent variables in powers of $r$.

Section 2 introduces the notation and derives the equations. §§3 and 4 contain certain asymptotic and approximate solutions. The numerical methods employed are discussed in §5, and in §6 the necessary extensions are given so that suction through the disk may be included. The results are given in the concluding sections.

## 2. The equations governing the motion

Cylindrical polar co-ordinates $(r, \phi, z)$ are used, with the disk in the plane $z = 0$, and the fluid occupies the region $z > 0$. Assuming the similarity solution, with the dependent variables in the form

$$u = r\Omega F(\zeta), \quad v = r\Omega G(\zeta), \quad w = (\nu\Omega)^{\frac{1}{2}} H(\zeta), \quad p/\rho = \nu\Omega P_0(\zeta) + \tfrac{1}{2}\kappa\Omega^2 r^2, \tag{1}$$

where $\kappa$ is a constant and $\zeta = z(\Omega/\nu)^{\frac{1}{2}}$, we find (see, for example, Schlichting 1955, p. 75) that the Navier–Stokes equations become

$$F'' = F^2 - G^2 + HF' + \kappa, \tag{2}$$

$$G'' = 2FG + HG', \tag{3}$$

$$P_0' = H'' - HH', \tag{4}$$

$$2F + H' = 0, \tag{5}$$

where a dash denotes differentiation with respect to $\zeta$. Equations (2), (3) and (5) must first be solved for the three velocity components $F$, $G$ and $H$, and then the pressure distribution can be found immediately since (4) integrates to give

$$P_0 = \Pi - 2F - \tfrac{1}{2}H^2.$$

The boundary conditions which must be satisfied at the disk are

$$u = 0, \quad v = r\Omega, \quad w = 0$$

and, in view of the similarity assumption (1), these are equivalent to

$$F(0) = 0, \quad G(0) = 1, \quad H(0) = 0. \tag{6}$$

At infinity, the fluid is rotating with uniform angular velocity $s\Omega$ and hence

$$F(\infty) = 0, \quad G(\infty) = s. \tag{7}$$

However, these conditions are not sufficient to determine the solution uniquely, for the constant $\kappa$ in equation (2) is still arbitrary. It is also necessary to assume that both $F'$ and $F''$ vanish at infinity, and thus it follows from (2) that

$$\kappa = s^2. \tag{8}$$

It will be seen, in fact, that the asymptotic solution which satisfies (7) automatically makes the velocity derivatives vanish at infinity, thus ruling out any possibility of a shear layer in the fluid at a great distance from the disk. Thus, the simplified set of Navier–Stokes equations to be solved is

$$F'' = F^2 - G^2 + HF' + s^2, \tag{9}$$

$$G'' = 2FG + HG', \tag{10}$$

$$2F + H' = 0, \tag{11}$$

subject to the boundary conditions (6) and (7).

When $s > 1$ it was found more convenient to use the angular velocity of the fluid at infinity as a reference velocity. Denoting this by $\omega$, and the angular velocity of the disk by $\sigma\omega$, a dimensionless co-ordinate is defined by $\xi = z(\omega/\nu)^{\frac{1}{2}}$. The velocity components are

$$u = r\omega\mathscr{F}(\xi), \quad v = r\omega\mathscr{G}(\xi), \quad w = (\nu\omega)^{\frac{1}{2}}\mathscr{H}(\xi)$$

and the differential equations become

$$\mathscr{F}'' = \mathscr{F}^2 - \mathscr{G}^2 + \mathscr{H}\mathscr{F}' + 1, \tag{12}$$

$$\mathscr{G}'' = 2\mathscr{F}\mathscr{G} + \mathscr{H}\mathscr{G}', \tag{13}$$

$$2\mathscr{F} + \mathscr{H}' = 0, \tag{14}$$

where dashes now denote differentiation with respect to $\xi$. The boundary conditions are

$$\left.\begin{aligned} &\mathscr{F}(0) = 0, \quad \mathscr{G}(0) = \sigma, \quad \mathscr{H}(0) = 0, \\ \text{and} \quad &\mathscr{F}(\infty) = 0, \quad \mathscr{G}(\infty) = 1. \end{aligned}\right\} \tag{15}$$

Since $\sigma\omega$ is the angular velocity of the disk, the range $0 < \sigma < 1$ corresponds to values of $s$ greater than 1. In particular, the value $\sigma = 0$ gives the problem already solved by Bödewadt, and his solution shows that boundary-layer effects extend out to about $\xi = 8$.

Various characteristics of the flow patterns can be computed, in particular the torque on the disk. Assuming for the present a finite radius $R$, and assuming the similarity solution is accurate over an appreciable part of the disk, we find that the moment is given by

$$\begin{aligned} M &= -2\pi \int_0^R r^2 \mu \left(\frac{\partial v}{\partial z}\right)_{z=0} dr \\ &= -\pi R^4 (\nu\Omega)^{\frac{1}{2}} G'(0). \end{aligned}$$

This quantity will be required subsequently in a discussion of the numerical solutions.

## 3. Asymptotic solutions

For large values of $z$, the flow is very nearly uniform solid rotation, and an asymptotic solution can easily be found. If we write

$$F = f(\zeta), \quad G = s+g(\zeta), \quad H = -c+h(\zeta), \tag{16}$$

where $c$ is the component of the velocity at $\infty$ in the axial direction *towards* the disk, and is a function of $s$, and then neglect squares and products of $f$, $g$ and $h$ and their derivatives, equations (9), (10) and (11) become

$$f'' = -2sg-cf', \tag{17}$$

$$g'' = +2sf-cg', \tag{18}$$

$$2f+h' = 0. \tag{19}$$

It follows from these equations that

$$f = e^{\lambda\zeta}\{A\cos\mu\zeta - B\sin\mu\zeta\} \tag{20}$$

and

$$g = e^{\lambda\zeta}\{B\cos\mu\zeta + A\sin\mu\zeta\}, \tag{21}$$

where

$$\lambda = \frac{1}{2}\left[-c-\left\{\frac{(c^4+64s^2)^{\frac{1}{2}}+c^2}{2}\right\}^{\frac{1}{2}}\right] \tag{22}$$

and

$$\mu = -\frac{1}{2}\left\{\frac{(c^4+64s^2)^{\frac{1}{2}}-c^2}{2}\right\}^{\frac{1}{2}}. \tag{23}$$

It should be noted that $\lambda$ is always negative, irrespective of the sign of $c$ and the magnitude of $s$. $A$ and $B$ are arbitrary constants whose values will be determined later for various values of $s$ by fitting the numerical solution.

These expressions display the oscillatory nature of the flow away from the disk and (23) shows the dependence of the wavelength on both the axial flow and the angular velocity of the fluid. These solutions are the leading terms in formal expansions for $F$ and $G$ in powers of $e^{\lambda\zeta}$. These expansions have been calculated by Bödewadt up to and including terms of order $e^{6\lambda\zeta}$. This was necessary in his work because the asymptotic solution was matched with a power series solution at $\zeta = 1$. However, in the present work the asymptotic solution is fitted to the machine solution at $\zeta = 6$, and then only as a final check on the solution. Consequently, only the first term is required. Asymptotic solutions of (12) and (13) will also be needed later: the derivation of these is precisely similar to the above results, and it is found that

$$\mathscr{F} = e^{\lambda_1\xi}\{\mathscr{A}\cos\mu_1\xi - \mathscr{B}\sin\mu_1\xi\} \tag{24}$$

and

$$\mathscr{G} = 1+e^{\lambda_1\xi}\{\mathscr{B}\cos\mu_1\xi + \mathscr{A}\sin\mu_1\xi\}, \tag{25}$$

where

$$\lambda_1 = \frac{1}{2}\left[-c-\left\{\frac{(c^4+64)^{\frac{1}{2}}+c^2}{2}\right\}^{\frac{1}{2}}\right] \tag{26}$$

and

$$\mu_1 = -\frac{1}{2}\left\{\frac{(c^4+64)^{\frac{1}{2}}-c^2}{2}\right\}^{\frac{1}{2}}. \tag{27}$$

As before, $-c$ is the limiting value of the axial component of velocity at infinity.

## 4. Approximate solutions

It will be seen in the next section that, for the numerical method employed, estimates of the velocity *derivatives* at the disk are needed. One way of obtaining these, of course, is to use the von Kármán integral method, but in practice it was found more accurate to linearize the equations about a state of solid rotation. When $s$ or $\sigma$ take values close to unity, the fluid at infinity is rotating with nearly the same angular velocity as the disk. Writing $\sigma = 1+\delta$ there are formal expansions for the functions $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ as follows

$$\mathscr{F} = \mathscr{F}_0+\delta\mathscr{F}_1+\delta^2\mathscr{F}_2+\ldots, \tag{28}$$

$$\mathscr{G} = \mathscr{G}_0+\delta\mathscr{G}_1+\delta^2\mathscr{G}_2+\ldots, \tag{29}$$

$$\mathscr{H} = \mathscr{H}_0+\delta\mathscr{H}_1+\delta^2\mathscr{H}_2+\ldots. \tag{30}$$

When $\delta = 0$, the state of solid rotation is recovered and thus

$$\mathscr{F}_0 = 0, \quad \mathscr{G}_0 = 1, \quad \mathscr{H}_0 = 0. \tag{31}$$

In view of the boundary conditions for $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$, the boundary conditions on $\mathscr{F}_n$, $\mathscr{G}_n$ and $\mathscr{H}_n$ are

$$\left.\begin{aligned} &\mathscr{F}_n(0) = \mathscr{F}_n(\infty) = 0 \quad (n = 1,2,3,\ldots),\\ &\mathscr{G}_n(0) = \mathscr{G}_n(\infty) = 0 \quad (n = 2,3,\ldots),\\ &\mathscr{G}_1(0) = 1, \quad \mathscr{G}_1(\infty) = 0,\\ &\mathscr{H}_n(0) = 0 \qquad\qquad\quad\; (n = 1,2,3,\ldots).\end{aligned}\right\} \tag{32}$$

If the above expansions are now substituted into (12) to (14) and coefficients of powers of $\delta$ are equated, sets of linear differential equations are obtained for the functions $\mathscr{F}_n$, $\mathscr{G}_n$, $\mathscr{H}_n$. The first few solutions are found to be

$$\left.\begin{aligned} \mathscr{F}_1 &= e^{-\xi}\sin\xi,\\ \mathscr{G}_1 &= e^{-\xi}\cos\xi,\\ \mathscr{H}_1 &= -1+e^{-\xi}(\sin\xi+\cos\xi);\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} \mathscr{F}_2 &= \frac{e^{-2\xi}}{10}-\frac{e^{-\xi}}{10}(\cos\xi-2\sin\xi)-\frac{\xi}{2}\sin\xi\, e^{-\xi},\\ \mathscr{G}_2 &= -\frac{e^{-2\xi}}{5}+\frac{e^{-\xi}}{10}(2\cos\xi+\sin\xi)-\frac{\xi}{2}\cos\xi\, e^{-\xi},\\ \mathscr{H}_2 &= \frac{3}{10}+\frac{e^{-2\xi}}{10}-\frac{\xi e^{-\xi}}{2}(\cos\xi+\sin\xi)-\frac{e^{-\xi}}{10}(4\cos\xi-3\sin\xi),\end{aligned}\right\} \tag{34}$$

and for the third set

$$\begin{aligned} \mathscr{F}_3+i\mathscr{G}_3 = {} & \frac{7(2i+7)}{400}\exp[-(1+i)\xi]+\frac{(2i-1)}{80}\exp[(i-3)\xi]+\frac{(i+3)}{100}\exp[-(i+3)\xi]\\ & -\frac{7(i+2)}{100}e^{-2\xi}+\frac{2i-1}{10}\xi e^{-2\xi}+i\frac{\xi^2}{8}\exp[-(1+i)\xi]-\frac{(1+i)}{80}\xi\exp[-(1+i)\xi]. \end{aligned} \tag{35}$$

The first approximation (33) was obtained by Squire (1953). It will be noticed that for large values of $\xi$, away from zeros of $\sin\xi$ and $\cos\xi$, the dominant terms in $\mathscr{F}_2$ and $\mathscr{G}_2$ are those involving $\xi$; similarly, in (35) the dominant term is of order $\xi^2$. It can be shown that the dominant term in $\mathscr{F}_n+i\mathscr{G}_n$ is

$$\frac{i(-1)^{n-1}\xi^{n-1}}{2^{n-1}(n-1)!}\exp\left[-(1+i)\xi\right]$$

and thus for large values of $\xi$, except near $\xi = n\pi$ and $(n+\frac{1}{2})\pi$, where $n$ takes positive integral values

$$\mathscr{F}+i\mathscr{G} \simeq i+\delta\exp\left[-(1+\tfrac{1}{2}\delta)\xi\right]\{\sin\xi+i\cos\xi\}. \tag{36}$$

The importance of these expansions lies in the fact that they provide fairly accurate values of $\mathscr{F}'(0)$ and $\mathscr{G}'(0)$ which are required in the numerical integration. From (33), (34) and (35)

$$\mathscr{F}'(0) = \delta+0{\cdot}1\delta^2+0{\cdot}0125\delta^3+O(\delta^4), \tag{37}$$

$$\mathscr{G}'(0) = -\delta-0{\cdot}2\delta^2+0{\cdot}0225\delta^3+O(\delta^4), \tag{38}$$

$$\mathscr{H}(\infty) = -\delta+0{\cdot}3\delta^2-0{\cdot}0875\delta^3+O(\delta^4). \tag{39}$$

Despite the fact that these expansions can only be expected to give reasonable accuracy for values of $|\delta|$ which are small compared with unity, on putting $\delta = -1$, so that the disk is at rest, these give the very accurate estimates

$$\mathscr{F}'(0) = -0{\cdot}9125, \quad \mathscr{G}'(0) = 0{\cdot}7775, \quad \mathscr{H}(\infty) = 1{\cdot}3875,$$

which may be compared with the final values, obtained numerically, namely

$$\mathscr{F}'(0) = -0{\cdot}9420, \quad \mathscr{G}'(0) = 0{\cdot}7729, \quad \mathscr{H}(\infty) = 1{\cdot}3696.$$

On the other hand, the form of equation (36) shows that trouble may be expected when $\delta \leqslant -2$ and this is borne out subsequently.

A similar linearization can be carried out for equations (9) to (11); writing $s = 1+\epsilon$, we find that

$$a \equiv F'(0) = -\epsilon-0{\cdot}4\epsilon^2+0{\cdot}0625\epsilon^3+O(\epsilon^4), \tag{40}$$

$$b \equiv G'(0) = \epsilon+0{\cdot}3\epsilon^2-0{\cdot}0475\epsilon^3+O(\epsilon^4), \tag{41}$$

$$-c \equiv H(\infty) = \epsilon-0{\cdot}2\epsilon^2+0{\cdot}0125\epsilon^3+O(\epsilon^4). \tag{42}$$

These are not so accurate as expressions (37), (38) and (39) but nevertheless provide suitable starting values for the subsequent calculations.

## 5. Numerical method of solution

The differential equations requiring numerical methods for their solution are those given in §2, (9)–(11). Solutions of these equations are required which satisfy the boundary conditions (6) and (7). For the purposes of computation on Pegasus, it is necessary to ensure that all quantities are less than unity in absolute value, so the following substitutions are made:

$$y_1 = \zeta 2^{-4}, \quad y_2 = F2^{-\alpha}, \quad y_3 = G2^{-\beta}, \quad y_4 = H2^{-\gamma}, \quad y_5 = F'2^{-\delta}, \quad y_6 = G'2^{-\epsilon}. \tag{43}$$

With these substitutions, the equations (9)–(11) may be rewritten as a system of *six first-order* differential equations. This reduction is performed because the standard Pegasus subroutine may then be used to solve them. The equations in question are:

$$\left.\begin{aligned} &y_1' = 2^{-4}, \quad y_2' = y_5\, 2^{\delta-\alpha}, \quad y_3' = y_6\, 2^{\epsilon-\beta}, \quad y_4' = -y_2\, 2^{\alpha+1-\gamma}, \\ &y_5' = y_2^2\, 2^{2\alpha-\delta} - y_3^2\, 2^{2\beta-\delta} + y_4 y_5\, 2^{\gamma} + s^2 2^{-\delta}, \quad y_6' = y_2 y_3\, 2^{\alpha+\beta+1-\epsilon} + y_4 y_6\, 2^{\gamma}. \end{aligned}\right\} \tag{44}$$

The first of these is used to keep track of how far the integration has proceeded. The associated boundary conditions are obtained by substitution of (43) into (6) and (7). Thus,

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 2^{-\beta}, \quad y_4(0) = 0;$$
$$y_2(\infty) = 0, \quad y_3(\infty) = s2^{-\beta}.$$

These six conditions are sufficient to define the problem mathematically but it must be remembered that, in §§2 and 3, the additional restrictions that $y_5(\infty)$ and $y_6(\infty)$ must be zero were imposed in order to obtain physically acceptable solutions.

In the numerical work, the values of the scaling factors were chosen to be $\alpha = 0$, $\beta = 2$, $\gamma = \delta = 3$ and $\epsilon = 4$. These choices, which were suggested by the solutions given by Cochran (1934) and Bödewadt (1940), proved to be satisfactory for all values of the parameter $s$.

Since the functions vary fairly rapidly for small values of $\zeta$ and more slowly for larger values, it was found convenient to use the Runge–Kutta method of integration as modified by Merson (1958). The modification suggested by Merson provides an automatic change of interval (either a decrease or an increase) when necessary.

In order to start the integration of the system (44) at $\zeta = 0$, it is necessary to estimate the values of $y_5(0)$ and $y_6(0)$; these are denoted by $a$ and $b$ (see (40) and (41)). Reasonably good estimates can be made using these equations in the cases when $s \simeq +1$. With these values, the system was integrated from zero to $\zeta = 12$. This was repeated using $a+\Delta a$, $b$, and $a$, $b+\Delta b$, as assumed values for $y_5(0)$ and $y_6(0)$. ($\Delta a$ and $\Delta b$ are small changes in $a$ and $b$.) In this way, three sets of values of $y_2(12)$ and $y_3(12)$ were obtained. These were used to evaluate

$$\frac{\partial y_2(12)}{\partial a}, \quad \frac{\partial y_2(12)}{\partial b}, \quad \frac{\partial y_3(12)}{\partial a} \quad \text{and} \quad \frac{\partial y_3(12)}{\partial b}.$$

Then, using the first-order terms in a Taylor expansion, it is possible to solve a pair of linear equations and to calculate $\delta a$ and $\delta b$, which are corrections to be applied to the original $a$ and $b$ in order to make $y_2(12)$ and $y_3(12)$ more nearly 0 and $s2^{-\beta}$, respectively.

Such a sequence of operations is called an iteration in the sequel. The iteration was repeated until $a$ and $b$ were obtained to an accuracy of about ten decimal places; in most cases this made $y_2(12)$ zero to an accuracy of about $\pm 10^{-6}$ and likewise $y_3(12)$ differed from $s2^{-\beta}$ by about $\pm 10^{-6}$.

This technique worked well for $s = +0{\cdot}9$ and $s = +0{\cdot}8$, but unfortunately the series (40) and (41) did not produce sufficiently accurate initial values of $a$ and $b$ for $s < +0{\cdot}8$. (The solutions diverged before the integration reached $\zeta = 12$.)

The obvious way to obtain more accurate values is to take more terms in the series but the algebra involved in obtaining further terms theoretically was prohibitive. Consequently, the coefficients of $\epsilon^4$ and $\epsilon^5$ in (40) and (41) were found numerically using the final, accurate values of $a$ and $b$ for the cases $s = +0{\cdot}9$ ($\equiv \epsilon = -0{\cdot}1$) and $s = +0{\cdot}8$ ($\equiv \epsilon = -0{\cdot}2$). The more accurate series obtained in this way are

$$\left.\begin{aligned} F'(0) &= -\epsilon - 0{\cdot}4\epsilon^2 + 0{\cdot}0625\epsilon^3 - 0{\cdot}01797\epsilon^4 + 0{\cdot}00748\epsilon^5 + O(\epsilon^6), \\ G'(0) &= +\epsilon + 0{\cdot}3\epsilon^2 - 0{\cdot}0475\epsilon^3 + 0{\cdot}01699\epsilon^4 - 0{\cdot}00894\epsilon^5 + O(\epsilon^6). \end{aligned}\right\} \tag{45}$$

Such a method enabled solutions to be obtained for $s = +0{\cdot}7$ and $+0{\cdot}6$, but for $s < +0{\cdot}6$ the initial values for $a$ and $b$ calculated from (45) allowed the exponentially increasing part of the solution to enter before the integration had reached $\zeta = 12$. Further coefficients in (45) could not be obtained using the previous method because all the significant figures cancelled and the resulting coefficients were very inaccurate.

Thus, a different approach was required for $s < +0{\cdot}6$. Von Kármán's solution was first improved by iterating with $s = 0$. Then the values of $a$ and $b$ so obtained were used to obtain an approximate solution in the case $s = +0{\cdot}1$. As was to be expected, divergence occurred for quite small values of $\zeta$ (of the order of 4). So the boundary conditions $y_2(\infty) = 0$ and $y_3(\infty) = s2^{-\beta}$ were satisfied at $\zeta = 3$ by iterating as before. The revised values of $a$ and $b$ so obtained enabled the integration to be extended to $\zeta = 4$ before divergence set in. In this way, iteration was applied at $\zeta = 4$ to make $y_2(4) = 0$ and $y_3(4) = s2^{-\beta}$. This process was repeated until $\zeta = 12$ was reached. At this stage, the changes in $a$ and $b$ from $\zeta = 11$ to $\zeta = 12$ were $O(10^{-9})$ and the calculation was stopped. This method was applied successfully for $s = +0{\cdot}1$ to $+0{\cdot}5$ so covering the range $0 \leqslant s \leqslant +1{\cdot}0$.

The range $s > +1$ was treated by considering the equations (12)–(15) written in terms of $\sigma = 1/s$. Obviously the original programme needed only trivial modification to do this. To obtain starting values for this set of equations, the series expansions (37) and (38) in terms of $\delta$ ($= \sigma - 1$) proved accurate enough to obtain solutions throughout the range $0 < \sigma < +1$ but two more terms of the series were calculated numerically as before. The results are:

$$\left.\begin{aligned} \mathscr{F}'(0) &= +\delta + 0{\cdot}1\delta^2 + 0{\cdot}0125\delta^3 - 0{\cdot}01263\delta^4 - 0{\cdot}00446\delta^5 + O(\delta^6), \\ \mathscr{G}'(0) &= -\delta - 0{\cdot}2\delta^2 + 0{\cdot}0225\delta^3 + 0{\cdot}00380\delta^4 - 0{\cdot}00115\delta^5 + O(\delta^6). \end{aligned}\right\} \tag{46}$$

So far in this section, attention has been confined to positive values of $s$ and $\sigma$. In order to obtain solutions for $-1 \leqslant s < 0$ the same technique as that used for *small* positive $s$ was employed. The values of $a$ and $b$ for $s = 0$ were used as initial estimates for $s = -0{\cdot}05$. This appeared to work satisfactorily for $s = -0{\cdot}05$, $-0{\cdot}10$ and $s = -0{\cdot}15$, but for $s = -0{\cdot}2$ the values of $F'(12)$ and $G'(12)$ were very different from zero. Thus, this solution (for $s = -0{\cdot}2$) was unacceptable on physical grounds.†. It was apparent that something peculiar was happening even

† The authors have been informed that in an unpublished investigation A. C. Browning found that the solution with $s = -0{\cdot}2$ is unacceptable on physical grounds. Browning also considered other values of $s$ or $\sigma$ and his solution for $\sigma = 0$ is reproduced in Schlichting (1955). Our results for this case agree better with those of Bödewadt (1940) than with those of Browning.

before the solution was printed out, because in going from the iteration at $\zeta = 11$ to that at $\zeta = 12$ the changes in $a$ and $b$ were *not* small.

Despite the fact that this solution was unsatisfactory on physical grounds, the method was continued to $s = -1{\cdot}0$ at which stage the values of $a$ and $b$ were very small, $O(10^{-4})$, but $F'(12)$ and $G'(12)$ were $O(1)$. Thus the region of large gradients, or boundary layer, seems to have moved away from the disk and large shears are appearing within the body of the fluid at the end of the integration range, where there is no solid boundary.

The solutions obtained for $s = -0{\cdot}05$, $-0{\cdot}1$ and $-0{\cdot}15$ appear satisfactory at first sight (that is, $F(12)$ and $G(12)$ are $O(10^{-4})$), but on closer examination they display several peculiar features. In the first place, it is shown in §2 that the torque on the disk is proportional to $|G'(0)| = |b|$. When $s = 0$, $|b| = 0{\cdot}615922$ and when $s > 0$ the value of $|b|$ is less than $0{\cdot}615922$, which is to be expected on physical grounds. Now it is also to be expected that $|b|$ will increase as $s$ becomes negative, since the disk is being rotated in the opposite sense to that of the fluid at infinity. It appears from the numerical solution that $|b|$ in fact decreases as $s$ becomes negative; for example, when $s = -0{\cdot}05$, $|b| = 0{\cdot}615676$, and when $s = -0{\cdot}1$, $|b| = 0{\cdot}608253$. The velocity components for these cases also attain their free-stream values at considerably greater distances from the disk than in the cases when $s$ is positive or zero. It may be that the situation would be clarified if the range of integration was extended; on the other hand, the changes in $a$ and $b$ in going from the iteration at $\zeta = 11$ to that at $\zeta = 12$ were small. An alternative boundary condition at infinity is to postulate that the velocity derivatives vanish there, but in this case the constant $\kappa$ in equation (2) is no longer known, and problems concerning the uniqueness of the solution then arise. The flow between two rotating disks is being currently investigated and it is hoped that this will throw further light on the problem.

It should perhaps be emphasized that great care was taken to ensure that a physically *sensible* solution had not been missed. For example, the value of $b$ which was guessed in the case $s = -0{\cdot}05$, was originally taken to be more negative than $-0{\cdot}615676$, but the 'solution' converged to the original one in all cases.

## 6. The problem with suction

Stuart (1954) has solved a similar problem with $s = 0$ and suction applied at the disk. As was to be expected, suction stabilizes the flow; that is, it reduces the magnitude of the radial and angular velocity components. The question thus naturally arises as to whether the application of suction to the disk when $s$ is negative, might prevent the boundary layer moving off the disk. Suction through the disk can be applied, in the present notation, by making $y_4(0)$ negative instead of zero. Following the notation of Stuart, a suction parameter $a^*$ can be introduced such that

$$H(0) = -a^*. \tag{47}$$

The equations governing the motion are, as before, (9)–(11), but the boundary conditions are (6) modified by (47) and (7).

The value $s = -1$ was chosen as a reasonably typical one and various values of the suction parameter $a^*$ were tried. It was found that the values 1·5, 1 and 0·8 all gave physically acceptable solutions and quite clearly larger values of $a^*$, corresponding to a stronger suction, would also be satisfactory. However, for $a^* = 0{\cdot}4$ no acceptable solution could be found: as before, the iterative process led to a solution for which the boundary layer is no longer attached to the disk, but appeared at the other end of the range of integration. In order to obtain a qualitative picture of the situation, the von Kármán integral method was employed in the following way. Equations (9) and (10) were integrated from $\zeta = 0$ to $\zeta = \delta$, subject to the boundary conditions

$$\left.\begin{aligned} F(0) &= 0, & G(0) &= 1, \\ F(\delta) &= 0, & G(\delta) &= s, \\ F'(\delta) &= 0, & G'(\delta) &= 0. \end{aligned}\right\} \tag{48}$$

On making use of equation (11), it follows that

$$-F'(0) = \int_0^\delta (3F^2 - G^2)\, d\zeta + s^2\delta \tag{49}$$

$$-G'(0) = 4\int_0^\delta FG\, d\zeta - 2s\int_0^\delta F\, d\zeta + a^*. \tag{50}$$

A quartic polynomial was assumed for $F$ and a cubic for $G$, and on putting $s = -1$, equations (49) and (50) reduce to

$$\frac{-\alpha}{\delta^2} = \frac{19\alpha^2}{210} + \frac{39}{70}, \tag{51}$$

$$\frac{3}{\delta^2} = \frac{13\alpha}{70} + \frac{a^*}{\delta}, \tag{52}$$

where $\alpha = \delta F'(0)$.

These equations give imaginary values of $\alpha$ if $a^* < 0{\cdot}257$, but real solutions for all values of the suction parameter greater than this. For very large values of $a^*$, (51) and (52) show that

$$F'(0) \simeq -\frac{117}{70a^*}, \quad G'(0) \simeq -2a^*, \tag{53}$$

and these are in qualitative agreement with the values obtained by expanding $F$ and $G$ in inverse powers of the suction parameter, as was done by Stuart for the case $s = 0$; the derivatives are

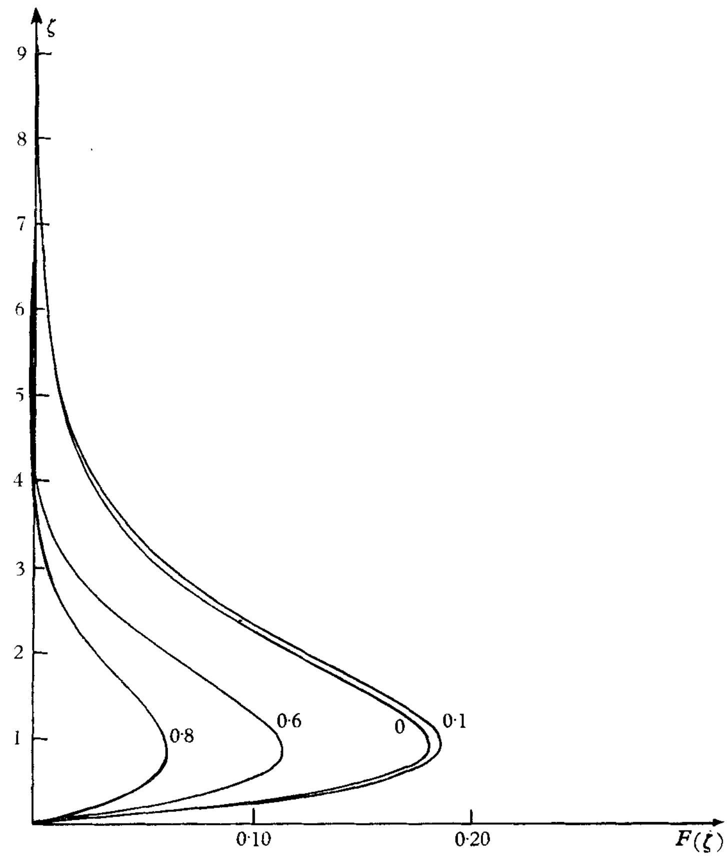
$$F'(0) = -\frac{2}{a^*} + O\left(\frac{1}{a^{*5}}\right); \quad G'(0) = -a^* + O\left(\frac{1}{a^{*3}}\right). \tag{54}$$

## 7. Results

Many results have already been mentioned in §§5 and 6, but some graphs and tables are presented here to emphasize the more important points. Figures 1–4 show the radial and transverse velocity profiles for different values of $s$ and $\sigma$. The velocity derivatives at the disk are given in Tables 1 and 2 for the indicated values of $s$ and $\sigma$, respectively. The constants describing the flow at a great distance from the disk are presented in Tables 3 and 4. These include the axial velocity of the fluid, the damping factors $\lambda$ and $\lambda_1$ defined in equations (22) and

(26), the wave-numbers $\mu$ and $\mu_1$ defined in (23) and (27), and the constants $A$, $B$, $\mathscr{A}$ and $\mathscr{B}$. It will be noticed that the values for $A$, $B$ and $c$ for $s = 0$ differ slightly from those obtained by Cochran.

An example of a flow in which suction at the disk is occurring and the fluid at infinity is rotating in the opposite direction to the disk has been computed;




Radial velocity profiles $0 \leqslant s < 1$

FIGURE 1. The function $F$ defined in equation (1), $0 \leqslant s \leqslant 1$. The value of $s$ is indicated in the figure. The radial velocity of the fluid is $r\Omega F$.

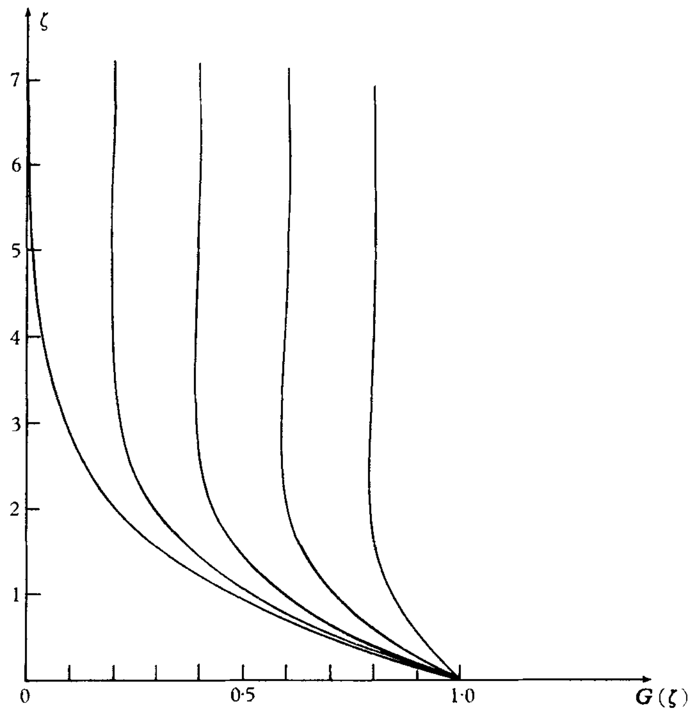
| $s$ | $F'(0) \equiv a$ | $G'(0) \equiv b$ |
|---|---|---|
| 0·0 | +0·510233 | −0·615922 |
| 0·1 | +0·513397 | −0·601554 |
| 0·2 | +0·501870 | −0·572080 |
| 0·4 | +0·439580 | −0·478673 |
| 0·6 | +0·331470 | −0·348434 |
| 0·8 | +0·183469 | −0·187591 |
| 0·9 | +0·095936 | −0·096951 |
| 1·0 | +0·0 | −0·0 |

TABLE 1. Values of the velocity derivatives $F'(0)$ and $G'(0)$ for various values of $s$.

complete numerical solutions for this case, and for the cases $s = 0{\cdot}6$ and $\sigma = 0{\cdot}8$ without suction, are being held at the Editor's office for consultation by interested people.

## 8. Conclusions

When the uniform rotation of the fluid at infinity is in the same sense as that of the disk, a physically acceptable solution exists for each value of $s$. In every case there is a boundary layer attached to the disk and in all cases, except one, the motion approaches the uniform rotation at infinity in an oscillatory fashion. The




Transverse velocity profiles $0 \leqslant s < 1$

FIGURE 2. The function $G$ defined in equation (1); the zonal velocity of the fluid is $r\Omega G$. These are all cases when the angular velocity of the disk is greater than that of the fluid at infinity.

| $\sigma$ | $\mathscr{F}'(0)$ | $\mathscr{G}'(0)$ |
|---|---|---|
| 0·0 | $-0{\cdot}941971$ | $+0{\cdot}772886$ |
| 0·1 | $-0{\cdot}844923$ | $+0{\cdot}718393$ |
| 0·2 | $-0{\cdot}751682$ | $+0{\cdot}658418$ |
| 0·4 | $-0{\cdot}569080$ | $+0{\cdot}522477$ |
| 0·6 | $-0{\cdot}385194$ | $+0{\cdot}366431$ |
| 0·8 | $-0{\cdot}196121$ | $+0{\cdot}191812$ |
| 0·9 | $-0{\cdot}099014$ | $+0{\cdot}097977$ |
| 1·0 | $-0{\cdot}0$ | $+0{\cdot}0$ |

TABLE 2. Values of the velocity derivatives $\mathscr{F}'(0)$ and $\mathscr{G}'(0)$ for various values of $\sigma$.

exception to this is the flow (for which $s = 0$) originally discussed by von Kármán in which the fluid at infinity is not rotating but moving in a purely axial direction. One unexpected feature of the solution is that the axial velocity at infinity for

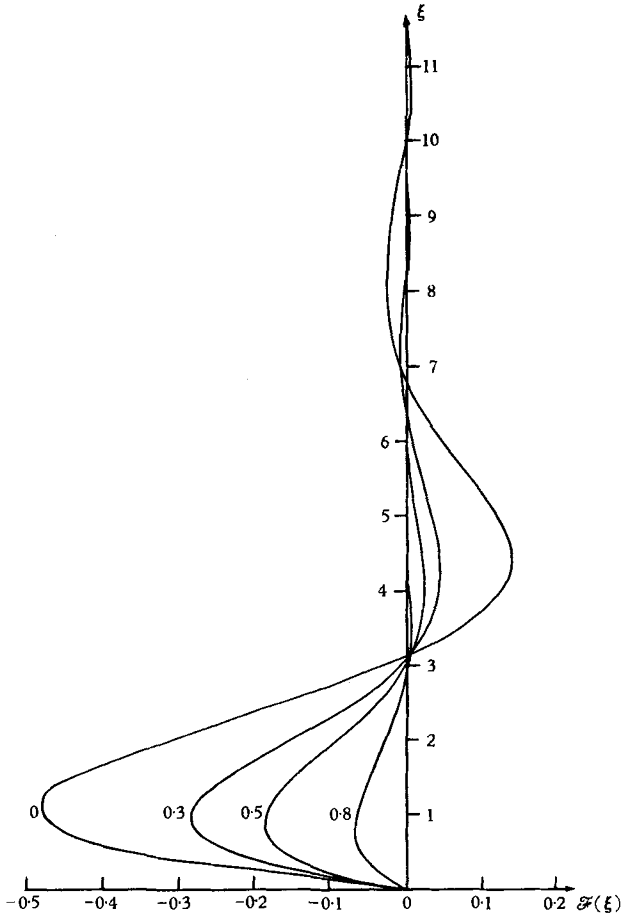



FIGURE 3. The function $\mathscr{F}$ defined in equation (12); the radial velocity of the fluid is $r\omega\mathscr{F}$, and the appropriate values of $\sigma$ are indicated on the figure.

| $s$ | $c \equiv -H(\infty)$ | $-\lambda$ | $-\mu$ | $A$ | $B$ |
|---|---|---|---|---|---|
| 0·0 | 0·88446 | 0·88446 | 0·0 | +0·91772 | +1·20211 |
| 0·1 | 0·91769 | 0·95931 | 0·19981 | +0·48981 | +1·32529 |
| 0·2 | 0·86175 | 0·99062 | 0·35730 | +0·19045 | +1·20306 |
| 0·4 | 0·65996 | 1·00683 | 0·59097 | +0·01320 | +0·75996 |
| 0·6 | 0·43077 | 1·00510 | 0·75977 | −0·00993 | +0·45192 |
| 0·8 | 0·20801 | 1·00146 | 0·89141 | −0·00332 | +0·21003 |
| 0·9 | 0·10201 | 1·00037 | 0·94800 | −0·00261 | +0·10431 |
| 1·0 | 0·0 | 1·0 | 1·0 | −0·0 | +0·0 |

TABLE 3. The flow at a great distance from the disk: these are the constants in (20) and (21) which give the approximate velocity distribution for large values of $\zeta$.

$s = 0\cdot1$ is stronger than the corresponding value for $s = 0$. The maximum value of the radial velocity is also greater in this case. This feature is also displayed by the approximate results of Fettis. In the range $0\cdot2 \leqslant s \leqslant 1$, the flow relative to the disk gets progressively weaker as $s$ increases to unity. No such anomalous

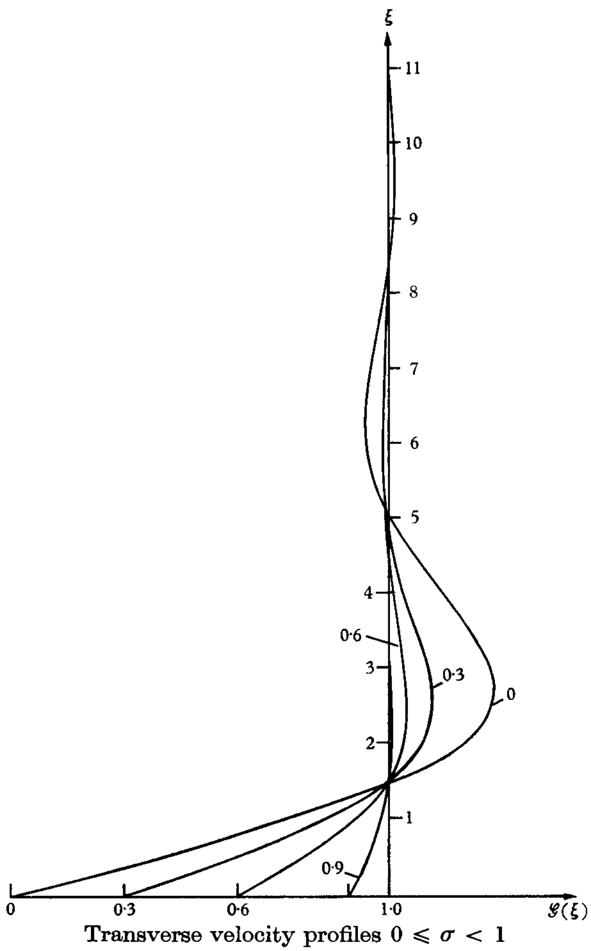



FIGURE 4. The function $\mathscr{G}$ defined in equation (12).

| $\sigma$ | $-c \equiv \mathscr{H}(\infty)$ | $-\lambda_1$ | $-\mu_1$ | $-\mathscr{A}$ | $-\mathscr{B}$ |
|---|---|---|---|---|---|
| 0·0 | 1·36961 | 0·43841 | 0·89031 | 0·23543 | 1·02911 |
| 0·1 | 1·19516 | 0·49529 | 0·91502 | 0·18303 | 0·88037 |
| 0·2 | 1·03080 | 0·55306 | 0·93593 | 0·13694 | 0·75559 |
| 0·4 | 0·72608 | 0·67043 | 0·96761 | 0·06787 | 0·55281 |
| 0·6 | 0·45378 | 0·78606 | 0·98721 | 0·02698 | 0·37316 |
| 0·8 | 0·21272 | 0·89647 | 0·99718 | 0·00749 | 0·19283 |
| 0·9 | 0·10309 | 0·94912 | 0·99934 | 0·00128 | 0·09882 |
| 1·0 | 0·0 | 1·0 | 1·0 | 0·0 | 0·0 |

TABLE 4. The flow at a great distance from the disk: these are the constants in (24) and (25) which give the approximate velocity distribution for large values of $\xi$.

behaviour is found in the range $1 \geqslant \sigma \geqslant 0$ and this may have some bearing on the fact that the expansions in powers of $\delta$ (equations (28)–(30)) are rather more accurate than the corresponding expansions in powers of $\epsilon$. In any case, the linearization technique is shown to give an accurate picture of the flow for quite a wide range of values of the expansion parameters $\delta$ and $\epsilon$.

When the fluid at infinity is rotating in the opposite sense to that of the disk, and $s \leqslant -0{\cdot}2$, then it appears that no steady solution is possible unless there is suction acting, which prevents the boundary layer from leaving the disk and attaching itself to the 'disturbing agency' at infinity. For each value of $s$ in this range there is probably a critical value of the suction parameter $a^*$, giving the minimum amount of suction needed to keep the boundary layer on the disk. All that can be said is that for the case $s = -1$, this critical value lies in the range $0{\cdot}8 > a^* > 0{\cdot}4$.

The situation as regards values of $s$ in the range $0 > s > -0{\cdot}2$ is not clear: solutions of the equations satisfying the postulated boundary conditions appear to exist, but they give anomalous values for the torque on the disk.

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